Noetherian rings with small profiles
(joint with S. R. López-Permouth)

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The profile of a ring

Definition

The (right) profile of a ring R is the lattice of all hereditary pretorsion classes $\mathscr{T}$ such that SSMod–$R \subseteq \mathscr{T}$, where SSMod–R denotes the class of all semisimple right R-modules. We will denote the profile of R by $\mathcal{P}(R)$. Any element of the profile is called a portfolio.

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This notion was first defined by Alahmadi, Alkan, and López-Permouth in 2010 in conjunction with the concept of injectivity domains.

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- $\mathcal{P}(R) = \{\text{Mod}\text{–}R\}$ if and only if R is semisimple Artinian.
- If $\mathcal{P}(R) = \{\text{SSMod}\text{–}R, \text{Mod}\text{–}R\}$, then we say that R has no right middle class (or briefly, R is a right NMC-ring).

Main references

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Some important hereditary pretorsion classes

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Mod-(R/A)

The category of right modules over R/A, where A is an ideal of R

Producing portfolios

Lemma (S. et al, 2020)

Let $\mathscr{T}$ be hereditary pretorsion classes of right R-modules. Then the class

$$\mathscr{T}^* = \{M \in Mod\text{–}R : M/\mathscr{T}(M) \text{ is semisimple and } \mathscr{T}\text{-torsion-free}\}$$

is a portfolio containing $\mathscr{T}$.

Use of Sing–R

Proposition (S. et al., 2020)

Let R be a right NMC-ring. Then every singular right R-module is semisimple. If, in addition, R is right nonsingular, then it is a right SI-ring.

Use of Sing–R

Theorem (S. et al., 2020)

A ring R is a right NMC-ring if and only if one of the following conditions holds for any hereditary pretorsion class $\mathscr{T}$ containing Sing–R:

(i) *Every $\mathscr{T}$-torsion module is semisimple, or*

(ii) *$\mathscr{T}$ is a torsion class and every $\mathscr{T}$-torsion-free module is semisimple and injective.*

Use of Sing–R

Corollary (S. et al., 2020)

Let R be any ring which is not semisimple Artinian. Then R is an indecomposable right NMC-ring if and only if SSMod–R is the unique coatom in the lattice of hereditary pretorsion classes.

Use of $\mathscr{B}$, the class of semiartinian modules

Proposition (Er et al., 2011; S. et al., 2020)

Let R be a right NMC-ring. Then R is either a right Noetherian, right V-ring or a right semiartinian ring.

Use of $\mathscr{N}$, the class of locally Noetherian modules

Proposition (Er et al., 2011; S. et al., 2020)

Let R be a right NMC-ring. Then R is either right Artinian or a right V-ring. Moreover, if $Z(R_R) \neq 0$, then R is right Artinian.

Proposition (S. et al., 2020)

If R is a right NMC-ring and if $R = A \times B$ is a ring decomposition, then either A or B is semisimple Artinian.

Use of the category Mod–(R/A)

Lemma (Er et al., 2011; S. et al., 2020)

Let R be a right NMC-ring. If A is a nonzero ideal of R, then either R/A or $R/ann_l A$ is a semisimple Artinian ring.

Use of $\sigma[M]$ category

Proposition (S. et al, 2020)

Let R be a right NMC-ring which is not semisimple Artinian. Then the following hold:

(i) *If $A \trianglelefteq R$ and R/A is not semisimple Artinian, then $R = A \oplus ann_l(A)$.*

(ii) *If A_1 and A_2 are nonzero two-sided ideals of R such that $A_1 \cap A_2 = 0$, then $R = A_i \oplus ann_l(A_i)$ for at least one $i = 1, 2$.*

(iii) *If R is indecomposable, then either $\mathrm{Soc}(R_R) = 0$ or $\mathrm{Soc}(R_R)$ is homogeneous and essential in R_R.*

The case when $\mathrm{Soc}(R_R) = 0$

Theorem (Er et al., 2011; S. et al., 2020)

Let R be an indecomposable ring with $\mathrm{Soc}(R_R) = 0$. Then R is a right NMC-ring if and only if it is Morita equivalent to a right SI-domain.

The Decomposition Theorem

Theorem (Er et al., 2011; S. et al., 2020)

A ring R is a right NMC-ring if and only if there exists a ring decomposition $R = A \times B$, where A is a semisimple Artinian ring and $B = 0$ or B is an indecomposable right NMC-ring.

Summary

R : right NMC-ring

$$R = \overbrace{S}^{\text{semisimple Artinian}} \times \overbrace{T}^{\text{indecomposable}}$$

$Z_r(T) \neq 0$:

- ●T is right Artinian
- ●$\mathrm{Soc}(T_T)$ is homogeneous
- ●$\mathrm{Soc}(T_T) = Z_r(T) = J(T)$

$Z_r(T) = 0$: T is right SI

- $\mathrm{Soc}(T_T) = 0$: T is Morita equivalent to a right PCI-domain.
- $\mathrm{Soc}(T_T) \neq 0$:
  - ○T is right Artinian or right V
  - ○$\mathrm{Soc}(T_T)$ is homogeneous
  - ○ $\mathrm{Soc}(T_T) \leq_{ess} T_T$

Existence problem of non-Noetherian NMC-rings

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Theorem (S. et al., 2020)

Let R be an indecomposable ring which is not right Noetherian. Then R is a right NMC-ring if and only if there exists a division ring D and a countably infinite dimensional right vector space V over D such that R is Morita equivalent to a subring T of $Q = End(V_D)$ with the following properties:
(i) T contains $\mathrm{Soc}(Q_Q)$, the set of all endomorphisms of V with finite rank,
(ii) $T/\mathrm{Soc}(Q_Q)$ is a division ring, and
(iii) for every $q \in Q \setminus \mathrm{Soc}(Q_Q)$, we have $Q = QqT$.

The case $\mathcal{P}(R) = \{\text{SSMod}–R, \text{Sing}–R, \text{Mod}–R\}$

Observation

If R is a right nonsingular ring such that $\mathcal{P}(R) = \{\text{SSMod}–R, \text{Sing}–R, \text{Mod}–R\}$, then every quasi-injective right R-module is either injective or semisimple.

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We call a ring right QIS if its every quasi-injective right module is either injective or semisimple.

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Definition

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- Every right QI-ring is right QIS.
- Every right NMC-ring is right QIS.

A classification of QIS-rings

Proposition

Let R be any ring. If $\mathrm{Soc}(R_R) \leq_e R_R$, *then R is a right NMC-ring if and only if it is a right QIS-ring.*

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Let R be any ring. If $\mathrm{Soc}(R_R) \leq_e R_R$, then R is a right NMC-ring if and only if it is a right QIS-ring.

Corollary

Let R be an indecomposable ring with nonzero right socle. Then R is right NMC if and only if it is right QIS.

A classification of QIS-rings

Theorem

Let R be any ring. Then R is a right QIS-ring if and only if one of the following conditions hold:

(i) *R is a right QI-ring.*

(ii) *R is a right NMC-ring.*

(iii) *There exists a ring decomposition $R = S \times T$ such that S is either zero or a semisimple Artinian ring and T is a right strongly prime right QIS-ring.*

Searching for an example

Definition (C. Faith, 1976)

We say that a ring R satisfies the restricted right socle condition if, whenever I is a proper essential right ideal of R, then R/I has at least one simple submodule.

Right SI-rings, hereditary prime Noetherian rings (HNP for short) and right NMC-rings satisfy this condition.

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Proposition

Let R be a right strongly prime ring with restricted right socle condition. Then R is a right QIS-ring if and only if
$\mathcal{P}(R) = \{SSMod\text{–}R, Sing\text{–}R, Mod\text{–}R\}$.

Searching for an example

Given a ring S and a right ideal A in S, one can form the largest subring of S, denoted $\mathbb{I}_S(A)$, containing A as a two-sided ideal. If A is such that $SA = S$ and $(S/A)_S$ is homogeneous semisimple, then the ring $\mathbb{I}_S(A)$ is called a basic idealizer.

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Proposition

Let S be an indecomposable right SI-ring with $\mathrm{Soc}(R_R) = 0$. If $R = \mathbb{I}_S(A)$ is a basic idealizer for a right ideal A of S, then R is a right QIS-ring which is neither right QI nor right NMC.

Searching for an example

Theorem

Let R be an HNP ring which is not simple. Then the following statements are equivalent:

(i) *R is a right QIS-ring which is neither right QI nor right NMC.*

(ii) *There is exactly one isomorphism class of non-injective simple right R-modules and if W is a non-injective simple module, then $E(W)$ is a uniserial module of length 2.*

(iii) *R is a basic idealizer ring from a right QI overring that is not simple Artinian.*

THANK YOU!